N14/5/MATHL/HP2/ENG/TZ0/XX/M

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MARKSCHEME

November 2014

MATHEMATICS

Higher Level

Paper 2

22 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking November 2014"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.

- If a part is completely wrong, stamp ***A0*** by the final answer.

- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, *N3*, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

> *Students must always use correct mathematical notation, not calculator notation.*

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. $\boldsymbol{n}_1 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ ***(A1)(A1)***

use of $\cos\theta = \frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ ***(M1)***

$\cos\theta = \frac{7}{\sqrt{21}\sqrt{19}} \left(= \frac{7}{\sqrt{399}} \right)$ ***(A1)(A1)***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$\theta = 69°$ ***A1***

Note: Award ***A1*** for $111°$.

Total [6 marks]

2. (a) $\mathrm{P}(X > x) = 0.99 \; (= \mathrm{P}(X < x) = 0.01)$ ***(M1)***

$\Rightarrow x = 54.6\,(\mathrm{cm})$ ***A1***

[2 marks]

(b) $\mathrm{P}(60.15 \le X \le 60.25)$ ***(M1)(A1)***

$= 0.0166$ ***A1***

[3 marks]

Total [5 marks]

3. use of $\mu = \frac{\sum_{i=1}^{k} f_i x_i}{n}$ to obtain $\frac{2 + x + y + 10 + 17}{5} = 8$ ***(M1)***

$x + y = 11$ ***A1***

EITHER

use of $\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}$ to obtain $\frac{(-6)^2 + (x-8)^2 + (y-8)^2 + 2^2 + 9^2}{5} = 27.6$ ***(M1)***

$(x-8)^2 + (y-8)^2 = 17$ ***A1***

OR

use of $\sigma^2 = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$ to obtain $\frac{2^2 + x^2 + y^2 + 10^2 + 17^2}{5} - 8^2 = 27.6$ ***(M1)***

$x^2 + y^2 = 65$ ***A1***

THEN

attempting to solve the two equations ***(M1)***

$x = 4$ and $y = 7$ (only as $x < y$) ***A1*** ***N4***

Note: Award ***A0*** for $x = 7$ and $y = 4$.

Note: Award ***(M1)A1(M0)A0(M1)A1*** for $x + y = 11 \Rightarrow x = 4$ and $y = 7$.

Total [6 marks]

4. **METHOD 1**

attempt to set up (diagram, vectors) ***(M1)***

correct distances $x = 15t,\ y = 20t$ ***(A1) (A1)***

the distance between the two cyclists at time t is $s = \sqrt{(15t)^2 + (20t)^2} = 25t$ (km) ***A1***

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\,h^{-1}})$ ***A1***

hence the rate is independent of time ***AG***

METHOD 2

attempting to differentiate $x^2 + y^2 = s^2$ implicitly ***(M1)***

$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2s\frac{\mathrm{d}s}{\mathrm{d}t}$ ***(A1)***

the distance between the two cyclists at time t is $\sqrt{(15t)^2 + (20t)^2} = 25t$ (km) ***(A1)***

$2(15t)(15) + 2(20t)(20) = 2(25t)\frac{\mathrm{d}s}{\mathrm{d}t}$ ***M1***

Note: Award ***M1*** for substitution of correct values into their equation involving $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\,h^{-1}})$ ***A1***

hence the rate is independent of time ***AG***

METHOD 3

$s = \sqrt{x^2 + y^2}$ ***(A1)***

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}}{\sqrt{x^2 + y^2}}$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to differentiate the expression for s.

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{(15t)(15) + (20t)(20)}{\sqrt{(15t)^2 + (20t)^2}}$ ***M1***

Note: Award ***M1*** for substitution of correct values into their $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\ h^{-1}})$ ***A1***

hence the rate is independent of time ***AG***

Total [5 marks]

5. (a) attempting to find a normal to π *eg* $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix}$ ***(M1)***

$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix} = 17 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***(A1)***

$\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***M1***

$2x - 2y + z = 4$ (or equivalent) ***A1***

[4 marks]

(b) $l_3 : \boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, t \in \mathbb{R}$ ***(A1)***

attempting to solve $\begin{pmatrix} 4+2t \\ -2t \\ 8+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$ for t *ie* $9t + 16 = 4$ for t ***M1***

$t = -\frac{4}{3}$ ***A1***

$\left(\frac{4}{3}, \frac{8}{3}, \frac{20}{3}\right)$ ***A1***

[4 marks]

Total [8 marks]

6. using $p(a) = -7$ to obtain $3a^3 + a^2 + 5a + 7 = 0$ ***M1A1***

$(a+1)(3a^2 - 2a + 7) = 0$ ***(M1)(A1)***

Note: Award ***M1*** for a cubic graph with correct shape and ***A1*** for clearly showing that the above cubic crosses the horizontal axis at (–1, 0) only.

$a = -1$ ***A1***

EITHER

showing that $3a^2 - 2a + 7 = 0$ has no real (two complex) solutions for *a* ***R1***

OR

showing that $3a^3 + a^2 + 5a + 7 = 0$ has one real (and two complex) solutions for *a* ***R1***

Note: Award ***R1*** for solutions that make specific reference to an appropriate graph.

Total [6 marks]

7. (a) using $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$ to form $\frac{a+2d}{a+6d} = \frac{a}{a+2d}$ ***(M1)***

$a(a+6d) = (a+2d)^2$ ***A1***

$2d(2d - a) = 0$ (or equivalent) ***A1***

since $d \neq 0 \Rightarrow d = \frac{a}{2}$ ***AG***

[3 marks]

(b) substituting $d = \frac{a}{2}$ into $a + 6d = 3$ and solving for *a* and *d* ***(M1)***

$a = \frac{3}{4}$ and $d = \frac{3}{8}$ ***(A1)***

$r = \frac{1}{2}$ ***A1***

$$\frac{n}{2}\left(2 \times \frac{3}{4} + (n-1)\frac{3}{8}\right) - \frac{3\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \geq 200$$ ***(A1)***

attempting to solve for *n* ***(M1)***

$n \geq 31.68\ldots$

so the least value of *n* is 32 ***A1***

[6 marks]

Total [9 marks]

8. (a) $3-\frac{t}{2}=0 \Rightarrow t=6\,(\text{s})$ ***(M1)A1***

[2 marks]

> **Note:** Award ***A0*** if either $t=-0.236$ or $t=4.24$ or both are stated with $t=6$.

(b) let d be the distance travelled before coming to rest

$d=\int_0^4 5-(t-2)^2\,\mathrm{d}t+\int_4^6 3-\frac{t}{2}\,\mathrm{d}t$ ***(M1)(A1)***

> **Note:** Award ***M1*** for two correct integrals even if the integration limits are incorrect. The second integral can be specified as the area of a triangle.

$d=\frac{47}{3}\,(=15.7)\,(\text{m})$ ***(A1)***

attempting to solve $\int_6^T\left(\frac{t}{2}-3\right)\mathrm{d}t=\frac{47}{3}$ (or equivalent) for T ***M1***

$T=13.9\,(\text{s})$ ***A1***

[5 marks]

Total [7 marks]

9. (a) each triangle has area $\frac{1}{8}x^2\sin\frac{2\pi}{n}$ (use of $\frac{1}{2}ab\sin C$) ***(M1)***

there are n triangles so $A=\frac{1}{8}nx^2\sin\frac{2\pi}{n}$ ***A1***

$$C=\frac{4\left(\frac{1}{8}nx^2\sin\frac{2\pi}{n}\right)}{\pi x^2}$$ ***A1***

so $C=\frac{n}{2\pi}\sin\frac{2\pi}{n}$ ***AG***

[3 marks]

(b) attempting to find the least value of n such that $\frac{n}{2\pi}\sin\frac{2\pi}{n}>0.99$ ***(M1)***

$n=26$ ***A1***

attempting to find the least value of n such that $\frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}>0.99$ ***(M1)***

$n=21$ (and so a regular polygon with 21 sides) ***A1***

> **Note:** Award ***(M0)A0(M1)A1*** if $\frac{n}{2\pi}\sin\frac{2\pi}{n}>0.99$ is not considered and $\frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}>0.99$ is correctly considered.
>
> Award ***(M1)A1(M0)A0*** for $n=26$.

[4 marks]

(c) **EITHER**

for even and odd values of n, the value of C seems to increase towards the limiting value of the circle $(C=1)$ *ie* as n increases, the polygonal regions get closer and closer to the enclosing circular region ***R1***

OR

the differences between the odd and even values of n illustrate that this measure of compactness is not a good one. ***R1***

[1 mark]

Total [8 marks]

SECTION B

10. (a) use of $A=\frac{1}{2}qr\sin\theta$ to obtain $A=\frac{1}{2}(x+2)(5-x)^2\sin 30^{\circ}$ ***M1***

$=\frac{1}{4}(x+2)(25-10x+x^2)$ ***A1***

$A=\frac{1}{4}(x^3-8x^2+5x+50)$ ***AG***

[2 marks]

(b) (i) $\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{4}\left(3x^2-16x+5\right)=\frac{1}{4}(3x-1)(x-5)$ ***A1***

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{4}\left(3\left(\frac{1}{3}\right)^2-16\left(\frac{1}{3}\right)+5\right)=0$ ***M1A1***

OR

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{4}\left(3\left(\frac{1}{3}\right)-1\right)\left(\left(\frac{1}{3}\right)-5\right)=0$ ***M1A1***

THEN

so $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***AG***

METHOD 2

solving $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ for x ***M1***

$-2<x<5\Rightarrow x=\frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***AG***

METHOD 3

a correct graph of $\frac{\mathrm{d}A}{\mathrm{d}x}$ versus x ***M1***

the graph clearly showing that $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ when $x=\frac{1}{3}$ ***AG***

[3 marks]

continued…

Question 10 continued

(c) (i) $\frac{d^2 A}{dx^2} = \frac{1}{2}(3x - 8)$ ***A1***

for $x = \frac{1}{3}$, $\frac{d^2 A}{dx^2} = -3.5 \, (< 0)$ ***R1***

so $x = \frac{1}{3}$ gives the maximum area of triangle PQR ***AG***

(ii) $A_{\max} = \frac{343}{27} \, (= 12.7) \, (\text{cm}^2)$ ***A1***

(iii) $\text{PQ} = \frac{7}{3}$ (cm) and $\text{PR} = \left(\frac{14}{3}\right)^2$ (cm) ***(A1)***

$\text{QR}^2 = \left(\frac{7}{3}\right)^2 + \left(\frac{14}{3}\right)^4 - 2\left(\frac{7}{3}\right)\left(\frac{14}{3}\right)^2 \cos 30^\circ$ ***(M1)(A1)***

$= 391.702\ldots$

$\text{QR} = 19.8 \, (\text{cm})$ ***A1***

[7 marks]

Total [12 marks]

11. (a) (i) $\mathrm{P}(X=0)=0.549\left(=\mathrm{e}^{-0.6}\right)$ ***A1***

(ii) $\mathrm{P}(X \geq 3)=1-\mathrm{P}(X \leq 2)$ ***(M1)***

$\mathrm{P}(X \geq 3)=0.0231$ ***A1***

[3 marks]

(b) **EITHER**

using $Y \sim \mathrm{Po}(3)$ ***(M1)***

OR

using $(0.549)^5$ ***(M1)***

THEN

$\mathrm{P}(Y=0)=0.0498\left(=\mathrm{e}^{-3}\right)$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) $\mathrm{P}(X=0)$ (most likely number of complaints received is zero) ***A1***

EITHER

calculating $\mathrm{P}(X=0)=0.549$ and $\mathrm{P}(X=1)=0.329$ ***M1A1***

OR

sketching an appropriate (discrete) graph of $\mathrm{P}(X=x)$ against x ***M1A1***

OR

finding $\mathrm{P}(X=0)=e^{-0.6}$ and stating that $\mathrm{P}(X=0)>0.5$ ***M1A1***

OR

using $\mathrm{P}(X=x)=\mathrm{P}(X=x-1)\times\dfrac{\mu}{x}$ where $\mu<1$ ***M1A1***

[3 marks]

(d) $\mathrm{P}(X=0)=0.8(\Rightarrow e^{-\lambda}=0.8)$ ***(A1)***

$\lambda=0.223\left(=\ln\dfrac{5}{4},=-\ln\dfrac{4}{5}\right)$ ***A1***

[2 marks]

Total [10 marks]

12. (a) $\mathrm{P}(\text{Ava wins on her first turn}) = \frac{1}{3}$ ***A1***

[1 mark]

(b) $\mathrm{P}(\text{Barry wins on his first turn}) = \left(\frac{2}{3}\right)^2$ ***(M1)***

$= \frac{4}{9} (= 0.444)$ ***A1***

[2 marks]

(c) $\mathrm{P}(\text{Ava wins in one of her first three turns})$

$= \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3}$ ***M1A1A1***

Note: Award ***M1*** for adding probabilities, award ***A1*** for a correct second term and award ***A1*** for a correct third term.
Accept a correctly labelled tree diagram, awarding marks as above.

$= \frac{103}{243} (= 0.424)$ ***A1***

[4 marks]

(d) $\mathrm{P}(\text{Ava eventually wins}) = \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \ldots$ ***(A1)***

using $S_\infty = \frac{a}{1-r}$ with $a = \frac{1}{3}$ and $r = \frac{2}{9}$ ***(M1)(A1)***

Note: Award ***(M1)*** for using $S_\infty = \frac{a}{1-r}$ and award ***(A1)*** for $a = \frac{1}{3}$ and $r = \frac{2}{9}$.

$= \frac{3}{7} (= 0.429)$ ***A1***

[4 marks]

Total [11 marks]

13. (a) attempting to use $V = \pi\int_a^b x^2 \mathrm{d}y$ ***(M1)***

attempting to express x^2 in terms of y *ie* $x^2 = 4(y+16)$ ***(M1)***

for $y = h$, $V = 4\pi\int_0^h y+16\,\mathrm{d}y$ ***A1***

$V = 4\pi\left(\frac{h^2}{2} + 16h\right)$ ***AG***

[3 marks]

(b) (i) **METHOD 1**

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}h} = 4\pi(h+16)$ ***(A1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$ ***M1A1***

Note: Award ***M1*** for substitution into $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$ ***AG***

METHOD 2

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (implicit differentiation) ***(M1)***

$\frac{-250\sqrt{h}}{\pi(h+16)} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (or equivalent) ***A1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$ ***M1A1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$ ***AG***

(ii) $\frac{\mathrm{d}t}{\mathrm{d}h} = -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$ ***A1***

$t = \int -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\,\mathrm{d}h$ ***(M1)***

$t = \int -\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\,\mathrm{d}h$ ***A1***

$t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$ ***AG***

continued…

Question 13 continued

(iii) **METHOD 1**

$$t = \frac{-4\pi^2}{250}\int_{48}^{0}\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$$ ***(M1)***

$t = 2688.756\ldots$ (s) ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$$t = \frac{-4\pi^2}{250}\left(\frac{2}{5}h^{\frac{5}{2}} + \frac{64}{3}h^{\frac{3}{2}} + 512h^{\frac{1}{2}}\right) + c$$

when $t = 0, h = 48 \Rightarrow c = 2688.756...\left(c = \frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}} + \frac{64}{3}\times 48^{\frac{3}{2}} + 512\times 48^{\frac{1}{2}}\right)\right)$ ***(M1)***

when $h = 0$, $t = 2688.756...\left(t = \frac{4\pi^2}{250}\left(\frac{2}{5}\times 48^{\frac{5}{2}} + \frac{64}{3}\times 48^{\frac{3}{2}} + 512\times 48^{\frac{1}{2}}\right)\right)$ (s) ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

[10 marks]

(c) **EITHER**

the depth stabilises when $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ *ie* $8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ ***R1***

attempting to solve $8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ for h ***(M1)***

OR

the depth stabilises when $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$ *ie* $\frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ ***R1***

attempting to solve $\frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ for h ***(M1)***

THEN

$h = 5.06$ (cm) ***A1***

[3 marks]

Total [16 marks]

14. (a) **METHOD 1**

squaring both equations ***M1***

$9\sin^2 B + 24\sin B\cos C + 16\cos^2 C = 36$ ***(A1)***

$9\cos^2 B + 24\cos B\sin C + 16\sin^2 C = 1$ ***(A1)***

adding the equations and using $\cos^2\theta + \sin^2\theta = 1$ to obtain

$9 + 24\sin(B+C) + 16 = 37$ ***M1***

$24(\sin B\cos C + \cos B\sin C) = 12$ ***A1***

$24\sin(B+C) = 12$ ***(A1)***

$\sin(B+C) = \frac{1}{2}$ ***AG***

METHOD 2

substituting for $\sin B$ and $\cos B$ to obtain

$\sin(B+C) = \left(\frac{6-4\cos C}{3}\right)\cos C + \left(\frac{1-4\sin C}{3}\right)\sin C$ ***M1***

$= \frac{6\cos C + \sin C - 4}{3}$ (or equivalent) ***A1***

substituting for $\sin C$ and $\cos C$ to obtain

$\sin(B+C) = \sin B\left(\frac{6-3\sin B}{4}\right) + \cos B\left(\frac{1-3\cos B}{4}\right)$ ***M1***

$= \frac{\cos B + 6\sin B - 3}{4}$ (or equivalent) ***A1***

Adding the two equations for $\sin(B+C)$:

$2\sin(B+C) = \frac{(18\sin B + 24\cos C) + (4\sin C + 3\cos B) - 25}{12}$ ***A1***

$\sin(B+C) = \frac{36+1-25}{24}$ ***(A1)***

$\sin(B+C) = \frac{1}{2}$ ***AG***

METHOD 3

substituting for $\sin B$ and $\sin C$ to obtain

$\sin(B+C) = \left(\frac{6-4\cos C}{3}\right)\cos C + \cos B\left(\frac{1-3\cos B}{4}\right)$ ***M1***

substituting for $\cos B$ and $\cos C$ to obtain

$\sin(B+C) = \sin B\left(\frac{6-3\sin B}{4}\right) + \left(\frac{1-4\sin C}{3}\right)\sin C$ ***M1***

Adding the two equations for $\sin(B+C)$:

$2\sin(B+C) = \frac{6\cos C + \sin C - 4}{3} + \frac{6\sin B + \cos B - 3}{4}$ (or equivalent) ***A1A1***

$2\sin\left(B+C\right)=\frac{(18\sin B+24\cos C)+(4\sin C+3\cos B)-25}{12}$ ***A1***

$\sin\left(B+C\right)=\frac{36+1-25}{24}$ ***(A1)***

$\sin\left(B+C\right)=\frac{1}{2}$ ***AG***

[6 marks]

(b) $\sin A=\sin\left(180^{\circ}-(B+C)\right)$ so $\sin A=\sin(B+C)$ ***R1***

$\sin(B+C)=\frac{1}{2}\Rightarrow\sin A=\frac{1}{2}$ ***A1***

$\Rightarrow A=30^{\circ}$ or $A=150^{\circ}$ ***A1***

if $A=150^{\circ}$, then $B<30^{\circ}$ ***R1***

for example, $3\sin B+4\cos C<\frac{3}{2}+4<6$, *ie* a contradiction ***R1***

only one possible value ($A=30^{\circ}$) ***AG***

[5 marks]

Total [11 marks]